

Killing–Yano symmetry of Kaluza–Klein black holes in five dimensions

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Abstract. Using a generalised Killing–Yano equation in the presence of torsion, spacetime metrics admitting a rank-2 generalised Killing–Yano tensor are investigated in five dimensions under the assumption that its eigenvector associated with the zero eigenvalue is a Killing vector field. It is shown that such metrics are classified into three types and the corresponding local expressions are given explicitly. It is also shown that they cover some classes of charged, rotating Kaluza–Klein black hole solutions of minimal supergravity and abelian heterotic supergravity.

1. Introduction

Killing–Yano tensors which were introduced by K. Yano [1] as a generalisation of Killing vectors to higher-rank antisymmetric tensors, have attracted the interests of many authors in the study of black hole physics as their presence endows black hole spacetimes with remarkable mathematical properties. For instance, B. Carter [2] studied a certain class of metrics including the Kerr metric as well as Kerr–Newman metric,

$$ds^2 = \frac{r^2 + p^2}{\mathcal{Q}(r)} dr^2 + \frac{r^2 + p^2}{\mathcal{P}(p)} dp^2 - \frac{\mathcal{Q}(r)}{r^2 + p^2} (d\tau + p^2 d\sigma)^2 + \frac{\mathcal{P}(p)}{r^2 + p^2} (d\tau - r^2 d\sigma)^2, \quad (1.1)$$

which is called Carter’s class. He demonstrated that for all the metrics of this class, both the Hamilton–Jacobi and Klein–Gordon equations can be solved by separation of variables. These separability structures are known to be deeply related to the existence of rank-2 Killing–Yano tensors. Moreover, W. Dietz and R. Rudiger [3] have shown that any four-dimensional metric admitting a rank-2 Killing–Yano tensor can be always written in the form of Carter’s class. Namely, in four dimensions spacetime metrics admitting a rank-2 Killing–Yano tensor involve many rotating black hole solutions of Einstein’s equations, e.g., Kerr and Kerr–Newman metrics, and due to the Killing–Yano symmetry, some test field equations such as Hamilton–Jacobi and Klein–Gordon equations are solvable by separation of variables. One of our tasks remaining in this direction is clarifying what happens in higher dimensions.

The investigation of Killing–Yano symmetry in higher-dimensional black hole spacetimes has just started in the last decade, e.g., see reviews [4–6]. It is known that in higher-dimensional vacuum solutions describing rotating black holes with spherical horizon topology [7–10], rank-2 closed conformal Killing–Yano tensors explain the separability of the Hamilton–Jacobi and Klein–Gordon equations on those spacetimes. Here, conformal Killing–Yano (CKY) tensors are antisymmetric tensors which were introduced by S. Tachibana [11] and T. Kashiwada [12] as a generalisation of conformal Killing vectors. However, CKY tensors are no longer useful to explain separability structure in higher-dimensional charged, rotating black hole spacetimes. Accordingly, a further generalisation of CKY tensors was introduced by the authors of [13, 14]. The generalised CKY tensors can be thought of as CKY tensors on spacetimes with a skew-symmetric torsion \mathbf{T} . The torsion is usually (not necessarily) identified with matter fluxes appearing in the theories. For instance, the five-dimensional gauged minimal supergravity black hole [15] admits a rank-2 generalised CKY tensor when the torsion is identified with the Hodge dual of the Maxwell field, $\mathbf{T} = *\mathbf{F}/\sqrt{3}$. It was also shown that the abelian heterotic supergravity black holes as well as their generalisation to higher dimensions [16–18] admit a rank-2 generalised CKY tensor. In this case, the torsion is identified with the 3-form field strength, $\mathbf{T} = \mathbf{H}$.

Compared to the asymptotically flat black holes, the separability structure of black strings or Kaluza–Klein black holes has not been studied very well. Since the inheritance

of separability by the uplift which is obvious for vacuum solutions is also expected in the presence of flux along the extra dimension, we are lead to the investigation of the generalised Killing–Yano symmetry in those spacetimes. In this paper, we thus elaborate the relationship between Killing–Yano symmetry and separability of the Hamilton–Jacobi and Klein–Gordon equations in charged, rotating Kaluza–Klein black hole spacetimes, especially in the five-dimensional minimal supergravity and abelian heterotic supergravity. In fact, it is shown that Killing–Yano symmetry of the Kaluza–Klein black holes are described by rank-2 generalised Killing–Yano tensors.

The presence of Killing–Yano symmetry itself is a strong enough restriction so that one can obtain explicit expressions of the metrics before imposing the dynamical equations. For example, the most general metric admitting a rank-2 closed CKY tensor was obtained in arbitrary dimensions [19, 20]. Even when it is not possible to write down the most general metric, this approach enables us to understand separability structure of various black hole spacetimes in a wider and unified framework [21, 22]. By making a suitable simplification, we indeed derive a class of five-dimensional metrics admitting a rank-2 generalised Killing–Yano tensor, which include and generalise the known examples of black strings and Kaluza–Klein black holes.

This paper is organised as follows: In Sec. 2, we first attempt to classify spacetime metrics admitting a rank-2 generalised Killing–Yano tensor in five dimensions under the assumption that there exists a particular Killing vector field. A large family of metrics is obtained. We find that resulting metrics are classified into three types, which we call type A, B and C, and some local expressions of the metrics are given explicitly. In Sec. 3, we consider the solution in the five-dimensional minimal supergravity, which is obtained as an uplift of the Kerr spacetime, and see that the metric falls into type A of the classified metrics. It is shown that a rank-2 generalised Killing–Yano tensor is responsible for separation of variables in the Hamilton–Jacobi and Klein–Gordon equations. Using the type A metric classified in Sec. 2, we construct the general solution in the five-dimensional minimal supergravity. In Sec. 4, we review the charged, rotating black string solution discovered by Mahapatra [23] in heterotic supergravity. We will see that the metric falls into type A again. The separability of the Hamilton–Jacobi and Klein–Gordon equations is also associated to a rank-2 generalised Killing–Yano tensor. Adopting the generalised Killing–Yano symmetry, we construct a class of charged, rotating Kaluza–Klein black hole solutions in the theory. Sec. 5 is devoted to summary and discussion.

2. Metrics admitting a rank-2 generalised Killing–Yano tensor in five dimensions

In this section, we attempt to classify spacetime metrics admitting a rank-2 generalised Killing–Yano tensor in five dimensions. Although we are interested in Lorentzian manifolds, for simplicity, the calculation in this section is carried out in Euclidean signature $(+, +, \dots, +)$. Applications to Kaluza–Klein spacetimes in mind, we assume

that there exists a particular Killing vector field that is an eigenvector of the generalised Killing–Yano tensor with zero eigenvalue, leaving more general investigations for a future work. In Sec. 2.1, we begin with reviewing the basics of the rank-2 generalised Killing–Yano tensors. Introducing canonical frames associated with such tensors, we derive the general forms of connection 1-forms in terms of the canonical frame in Sec. 2.2. The computation there is performed by exploiting the technique of [22]. Furthermore, we proceed to restrict the forms of the connection 1-forms by imposing integrability conditions, so that commutation relations among canonical basis vectors are obtained in Sec. 2.3. Finally, solving the commutation relations in Sec. 2.4–2.6, some local expressions of the metrics are given explicitly. In the process of solving the commutation relations, we find that resulting metrics are classified into three types, which we call type A, B and C.

2.1. Basics

Let (M, \mathbf{g}) be a five-dimensional Riemannian manifold and $\{\mathbf{e}_a\}$ be an orthonormal frame. Throughout the article, Latin indices a, b, \dots range from 1 to 5. Greek letters μ, ν, \dots will later be used to denote two-dimensional eigenspaces of non-zero eigenvalues of a 2-form. The dual frame $\{\mathbf{e}^a\}$ satisfies $\mathbf{e}_a \lrcorner \mathbf{e}^b = \delta_a^b$ where \lrcorner represents the inner product. A p -form \mathbf{k} is written as

$$\mathbf{k} = \frac{1}{p!} k_{a_1 \dots a_p} \mathbf{e}^{a_1} \wedge \mathbf{e}^{a_p} , \quad k_{[a_1 \dots a_p]} = k_{a_1 \dots a_p} . \quad (2.1)$$

For a 2-form \mathbf{f} , a rank-2 Killing–Yano tensor, introduced by [1], is subject to the equation

$$\nabla_a f_{bc} + \nabla_b f_{ca} = 0 , \quad (2.2)$$

where ∇_a is the Levi-Civita connection. In the presence of a skew-symmetric torsion, $T_{[abc]} = T_{abc}$, a connection ∇_a^T is defined by

$$\nabla_a^T X^b = \nabla_a X^b + \frac{1}{2} T_{ac}{}^b X^c . \quad (2.3)$$

By replacing the connections ∇_a in (2.2) with ∇_a^T , rank-2 generalised Killing–Yano (GKY) tensors are defined [13] by

$$\nabla_a^T f_{bc} + \nabla_b^T f_{ac} = 0 . \quad (2.4)$$

For a rank- p GKY tensor, the Hodge dual gives a rank- $(D - p)$ generalised closed conformal Killing–Yano (GCCKY) tensor in D dimensions [21]. In $D = 5$, the Hodge dual $\mathbf{h} = *\mathbf{f}$ of a rank-2 GKY tensor \mathbf{f} is a rank-3 GCCKY tensor obeying

$$\nabla_a^T h_{bcd} = g_{ab} \xi_{cd} + g_{ac} \xi_{db} + g_{ad} \xi_{bc} , \quad (2.5)$$

where

$$\xi_{ab} = \frac{1}{3} \nabla^T c h_{cab} \quad (2.6)$$

is called an associated 2-form of \mathbf{h} . Eq. (2.5) implies that

$$\nabla_{[a}^T h_{bcd]} = 0 \ , \quad \nabla^{Tb} \xi_{ba} = 0 \ . \quad (2.7)$$

In general, rank-2 GKY tensors are related to separation constants of Hamilton–Jacobi equations for geodesics. For a rank-2 GKY tensor \mathbf{f} , the corresponding separation constant $\kappa^{(HJ)}$ is given by

$$\kappa^{(HJ)} = K_{ab} \Pi^a \Pi^b \ , \quad (2.8)$$

where Π^a is the canonical momentum associated with the geodesic and

$$K_{ab} = f_{ac} f_b^c \quad (2.9)$$

is a rank-2 Killing–Stäckel tensor satisfying $\nabla_{(a} K_{bc)} = 0$. The separation constant of the Klein–Gordon equation $\kappa^{(KG)}$ appears as an eigenvalue of a symmetry operator [24]

$$\hat{K} \equiv \nabla_a K^{ab} \nabla_b \quad (2.10)$$

that satisfies $[\hat{K}, \square] = 0$ where \square is the scalar wave operator $\square \equiv g^{ab} \nabla_a \nabla_b$. That is,

$$\hat{K} \Psi = \kappa^{(KG)} \Psi \ . \quad (2.11)$$

2.2. General forms of the connection 1-forms

Let us consider a rank-3 GCCKY tensor \mathbf{h} in five dimensions, which is equivalent to considering a rank-2 GKY tensor \mathbf{f} . Then one can always find an orthonormal frame $\{\mathbf{e}^a\} = \{\mathbf{e}^\mu, \mathbf{e}^{\hat{\mu}} = \mathbf{e}^{2+\mu}, \mathbf{e}^0 = \mathbf{e}^5\}$, $\mu = 1, 2$ such that a metric \mathbf{g} and a rank-3 GCCKY tensor \mathbf{h} are simultaneously written in the form

$$\mathbf{g} = \sum_{\mu=1}^2 (\mathbf{e}^\mu \otimes \mathbf{e}^\mu + \mathbf{e}^{\hat{\mu}} \otimes \mathbf{e}^{\hat{\mu}}) + \mathbf{e}^0 \otimes \mathbf{e}^0 \ , \quad (2.12)$$

$$\mathbf{h} = \sum_{\mu=1}^2 x_\mu \mathbf{e}^\mu \wedge \mathbf{e}^{\hat{\mu}} \wedge \mathbf{e}^0 \ , \quad (2.13)$$

where x_μ are called the eigenvalues of \mathbf{h} . The rank-3 GCCKY tensor is said to be non-degenerate if its eigenvalues x_μ are non-vanishing functions with $x_1 \neq x_2$. Since there are still degrees of freedom under rotation in each $(e_\mu, e_{\hat{\mu}})$ -plane, the orthonormal frame is fixed completely by introducing a 1-form $\boldsymbol{\eta}$ as

$$\boldsymbol{\eta} = -\mathbf{e}_0 \lrcorner \boldsymbol{\xi} = \sqrt{Q_1} \mathbf{e}^{\hat{1}} + \sqrt{Q_2} \mathbf{e}^{\hat{2}} \ , \quad (2.14)$$

where $\boldsymbol{\xi}$ is the associated 2-form introduced in (2.6) and Q_1 and Q_2 are unknown functions. This means that we used the remaining rotations so as to set $\xi_{10} = \xi_{20} = 0$. The fixed orthonormal frame is called a canonical frame. With respect to the canonical frame, the rank-2 GKY tensor \mathbf{f} and rank-2 Killing–Stäckel tensor \mathbf{K} , given by (2.9), are written as

$$\mathbf{f} = x_2 \mathbf{e}^1 \wedge \mathbf{e}^{\hat{1}} + x_1 \mathbf{e}^2 \wedge \mathbf{e}^{\hat{2}} \ , \quad (2.15)$$

$$\mathbf{K} = x_2^2 (\mathbf{e}^1 \otimes \mathbf{e}^1 + \mathbf{e}^{\hat{1}} \otimes \mathbf{e}^{\hat{1}}) + x_1^2 (\mathbf{e}^2 \otimes \mathbf{e}^2 + \mathbf{e}^{\hat{2}} \otimes \mathbf{e}^{\hat{2}}) \ . \quad (2.16)$$

The GCCKY equation (2.5) can be thought of as relating components of connection 1-forms ω^a_b to the eigenvalues x_μ of the GCCKY tensor \mathbf{h} and their derivatives. Exploiting the technique of [22] which was developed for rank-2 GCCKY tensors, we now apply it to rank-3 GCCKY tensors in five dimensions. Thus a similar calculation leads us to the following results:

$$\omega^\mu{}_\nu = -\frac{x_\nu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2}e^\mu - \frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2}e^\nu - \frac{x_\nu\xi_{\mu\hat{\nu}} + x_\mu\xi_{\nu\hat{\mu}}}{x_\mu^2 - x_\nu^2}e^0, \quad (2.17)$$

$$\omega^\mu{}_{\hat{\mu}} = -\frac{1}{\sqrt{Q_\mu}}\frac{x_\mu Q_\nu}{x_\mu^2 - x_\nu^2}e^{\hat{\mu}} + \frac{x_\mu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2}e^{\hat{\nu}} \quad (2.18)$$

$$-\frac{\sqrt{Q_\nu}}{\sqrt{Q_\mu}}\frac{x_\nu\xi_{\mu\nu} + x_\mu\xi_{\hat{\mu}\hat{\nu}}}{x_\mu^2 - x_\nu^2}e^0 + \sum_a \frac{\kappa_{a\mu}}{\sqrt{Q_\mu}}e^a, \quad (2.19)$$

$$\omega^\mu{}_{\hat{\nu}} = \frac{x_\mu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2}e^{\hat{\mu}} - \frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2}e^{\hat{\nu}} + \frac{x_\nu\xi_{\mu\nu} + x_\mu\xi_{\hat{\mu}\hat{\nu}}}{x_\mu^2 - x_\nu^2}e^0, \quad (2.20)$$

$$\omega^{\hat{\mu}}{}_{\hat{\nu}} = -\frac{x_\mu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2}e^\mu - \frac{x_\nu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2}e^\nu - \frac{x_\mu\xi_{\mu\hat{\nu}} + x_\nu\xi_{\nu\hat{\mu}}}{x_\mu^2 - x_\nu^2}e^0, \quad (2.21)$$

$$\omega^\mu{}_0 = \frac{\xi_{\nu\hat{\nu}}}{x_\nu}e^\mu - \frac{\xi_{\mu\hat{\nu}}}{x_\nu}e^\nu + \frac{\xi_{\mu\nu}}{x_\nu}e^{\hat{\nu}}, \quad (2.22)$$

$$\omega^{\hat{\mu}}{}_0 = -\frac{\xi_{\hat{\mu}\hat{\nu}}}{x_\nu}e^\nu + \frac{\xi_{\nu\hat{\nu}}}{x_\nu}e^{\hat{\mu}} - \frac{\xi_{\nu\hat{\mu}}}{x_\nu}e^{\hat{\nu}}, \quad (2.23)$$

where the symbols κ_{ab} are defined as

$$\kappa_{ab} = \mathbf{e}_b \lrcorner \nabla_{\mathbf{e}_a}^T \boldsymbol{\eta}. \quad (2.24)$$

2.3. Integrability conditions

To obtain the information about second derivatives κ_{ab} , we consider the integrability conditions of the Killing–Yano equation (2.5),

$$\begin{aligned} & -R^{Tf}{}_{ade}h_{fbc} - R^{Tf}{}_{bde}h_{fca} - R^{Tf}{}_{cde}h_{fab} \\ & = g_{ae}\nabla_d^T\xi_{bc} + g_{be}\nabla_d^T\xi_{ca} + g_{ce}\nabla_d^T\xi_{ab} - (d \leftrightarrow e) \\ & + T_{ade}\xi_{bc} + T_{bde}\xi_{ca} + T_{cde}\xi_{ab}, \end{aligned} \quad (2.25)$$

where $R^{Ta}{}_{bcd}$ are components of the Riemann tensor with respect to ∇_a^T defined by

$$(\nabla_a^T\nabla_b^T - \nabla_b^T\nabla_a^T + T_{ab}{}^d\nabla_d^T)Z_c \equiv -R^{Td}{}_{cab}Z_d. \quad (2.26)$$

The components do not satisfy the Bianchi identities $R^{Ta}{}_{[bcd]} = 0$, while they have the following symmetries among their indices:

$$R^T{}_{bacd} = -R^T{}_{abcd}, \quad R^T{}_{abdc} = -R^T{}_{abcd}. \quad (2.27)$$

The general form of the integrability conditions is too complicated to solve analytically since it contains many coupled, nonlinear partial differential equations. Therefore, for simplicity, we impose an assumption that \mathbf{e}_0 is a Killing vector field. This is motivated by the fact that the known example which admits a rank-3 GCCKY

tensor in the literature [25] satisfies this assumption. It leads us to vanishing of many components of the associated 2-form $\boldsymbol{\xi}$ and the torsion \boldsymbol{T} . In fact, the Killing equation $\nabla_a(\mathbf{e}^0)_b + \nabla_b(\mathbf{e}^0)_a = 0$ implies

$$\xi_{\mu\nu} = \xi_{\mu\hat{\mu}} = \xi_{\mu\hat{\nu}} = \xi_{\hat{\mu}\hat{\nu}} = 0 , \quad (2.28)$$

and hence it follows from the integrability conditions (2.25) that

$$\begin{aligned} T_{\mu\nu\hat{\nu}} &= T_{\mu\nu 0} = T_{\mu\hat{\nu}0} = T_{\hat{\mu}\hat{\nu}0} = 0 , \\ T_{\mu\hat{\mu}0} &= -\frac{\kappa_{0\mu}}{\sqrt{Q_\mu}} , \quad T_{\mu\hat{\mu}\hat{\nu}} = -\frac{\kappa_{\hat{\nu}\mu}}{\sqrt{Q_\mu}} . \end{aligned} \quad (2.29)$$

The commutators now simplify to

$$\begin{aligned} [\mathbf{e}_\mu, \mathbf{e}_\nu] &= -\frac{x_\nu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} \mathbf{e}_\mu - \frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2} \mathbf{e}_\nu , \\ [\mathbf{e}_\mu, \mathbf{e}_{\hat{\mu}}] &= K_\mu \mathbf{e}_\mu + L_\mu \mathbf{e}_{\hat{\mu}} + M_\mu \mathbf{e}_{\hat{\nu}} - T_{\mu\hat{\mu}0} \mathbf{e}_0 , \\ [\mathbf{e}_\mu, \mathbf{e}_{\hat{\nu}}] &= -\frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2} \mathbf{e}_{\hat{\nu}} , \\ [\mathbf{e}_{\hat{\mu}}, \mathbf{e}_{\hat{\nu}}] &= 0 , \quad [\mathbf{e}_\mu, \mathbf{e}_0] = 0 , \quad [\mathbf{e}_{\hat{\mu}}, \mathbf{e}_0] = 0 , \end{aligned} \quad (2.30)$$

where

$$\begin{aligned} K_\mu &= \frac{\kappa_{\mu\mu}}{\sqrt{Q_\mu}} , \quad L_\mu = -\frac{1}{\sqrt{Q_\mu}} \left(\frac{x_\mu Q_\nu}{x_\mu^2 - x_\nu^2} - \kappa_{\hat{\mu}\mu} \right) , \\ M_\mu &= \frac{2x_\mu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} - T_{\mu\hat{\mu}\hat{\nu}} . \end{aligned} \quad (2.31)$$

The classification problem under the assumption that \mathbf{e}_0 is a Killing vector field reduces to solving the commutators (2.30). The integrability conditions, e.g., the Jacobi identities, which have not been satisfied yet, give rise to two algebraic equations

$$M_1 K_2 = 0 , \quad M_2 K_1 = 0 , \quad (2.32)$$

and a system of coupled, nonlinear partial differential equations

$$\begin{aligned} \mathbf{e}_\nu(K_\mu) &= \frac{x_\nu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} K_\mu , \\ \mathbf{e}_\nu(L_\mu) &= \frac{x_\nu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} L_\mu - M_\mu M_\nu - \frac{2x_\mu x_\nu \sqrt{Q_\mu} \sqrt{Q_\nu}}{(x_\mu^2 - x_\nu^2)^2} , \\ \mathbf{e}_\nu(M_\mu) &= \left(\frac{2x_\nu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} - L_\nu \right) M_\mu , \\ \mathbf{e}_\nu(T_{\mu\hat{\mu}0}) &= \frac{2x_\nu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2} T_{\mu\hat{\mu}0} - M_\mu T_{\nu\hat{\nu}0} , \end{aligned} \quad (2.33)$$

and

$$\begin{aligned} \mathbf{e}_{\hat{\nu}}(K_\mu) &= 0 , \quad \mathbf{e}_{\hat{\nu}}(L_\mu) = 0 , \quad \mathbf{e}_{\hat{\nu}}(M_\mu) = 0 , \quad \mathbf{e}_{\hat{\nu}}(T_{\mu\hat{\mu}0}) = 0 , \\ \mathbf{e}_0(K_\mu) &= 0 , \quad \mathbf{e}_0(L_\mu) = 0 , \quad \mathbf{e}_0(M_\mu) = 0 , \quad \mathbf{e}_0(T_{\mu\hat{\mu}0}) = 0 . \end{aligned} \quad (2.34)$$

Since at least there exist orthonormal frames satisfying (2.30) if its integrability conditions hold, the algebraic equations (2.32) imply that there are three types of the

solutions: A) $K_1 = K_2 = 0$, B) $M_1 = M_2 = 0$ and C) $K_1 = M_1 = 0$ or $K_2 = M_2 = 0$. For each type, we are able to find large families of solutions of the remaining partial differential equations.

2.4. Type A metric

2.4.1. Type AI

Firstly, we consider $K_1 = K_2 = 0$ case. In this case, we find the canonical frame $\{\mathbf{e}_a\}$ satisfying the commutators (2.30) as

$$\begin{aligned} \mathbf{e}_\mu &= \sqrt{\frac{X_\mu}{x_\mu^2 - x_\nu^2}} \frac{\partial}{\partial x_\mu}, \quad \mathbf{e}_0 = \frac{\partial}{\partial \psi}, \\ \mathbf{e}_{\hat{\mu}} &= \frac{1}{f_\mu \sqrt{(x_\mu^2 - x_\nu^2) X_\mu}} \left((x_\mu^2 + N_\mu) \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} - \lambda N_\mu \frac{\partial}{\partial \psi} \right), \end{aligned} \quad (2.35)$$

where X_μ , N_μ and f_μ are unknown functions of one variable x_μ and λ is an arbitrary constant. Rewriting (x_1, x_2) by (x, y) , the corresponding metric is written by

$$\begin{aligned} \mathbf{g} &= \frac{x^2 - y^2}{X(x)} dx^2 + \frac{y^2 - x^2}{Y(y)} dy^2 + (d\psi + \lambda \mathbf{W}_1)^2 \\ &+ \frac{f_1(x)^2 X(x)}{x^2 - y^2} (d\tau + y^2 d\sigma - \mathbf{W}_1)^2 + \frac{f_2(y)^2 Y(y)}{y^2 - x^2} (d\tau + x^2 d\sigma - \mathbf{W}_1)^2, \end{aligned} \quad (2.36)$$

where

$$\begin{aligned} \mathbf{W}_1 &= \frac{N_1(x)}{(x^2 - y^2)\Phi} (d\tau + y^2 d\sigma) + \frac{N_2(y)}{(y^2 - x^2)\Phi} (d\tau + x^2 d\sigma), \\ \Phi &= 1 + \frac{N_1(x)}{x^2 - y^2} + \frac{N_2(y)}{y^2 - x^2}. \end{aligned} \quad (2.37)$$

The metric contains 6 unknown functions of one variable, $X(x)$, $Y(y)$, $N_1(x)$, $N_2(y)$, $f_1(x)$ and $f_2(y)$. Since the metric components are independent of the coordinates τ , σ and ψ , $\partial/\partial\tau$, $\partial/\partial\sigma$ and $\partial/\partial\psi$ are three Killing vector fields. Then the torsion is given by

$$\begin{aligned} \mathbf{T} &= \left[\frac{2x}{x^2 - y^2} - \frac{f_2(y)}{f_1(x)} \left(\frac{2x}{x^2 - y^2} + \frac{\partial \ln \Phi}{\partial x} \right) \right] \sqrt{\frac{Y(y)}{y^2 - x^2}} \mathbf{e}^1 \wedge \mathbf{e}^{\hat{1}} \wedge \mathbf{e}^{\hat{2}} \\ &+ \left[\frac{2y}{y^2 - x^2} - \frac{f_1(x)}{f_2(y)} \left(\frac{2y}{y^2 - x^2} + \frac{\partial \ln \Phi}{\partial y} \right) \right] \sqrt{\frac{X(x)}{x^2 - y^2}} \mathbf{e}^2 \wedge \mathbf{e}^{\hat{2}} \wedge \mathbf{e}^{\hat{1}} \\ &+ \frac{\lambda}{f_1(x)} \frac{\partial \ln \Phi}{\partial x} \mathbf{e}^1 \wedge \mathbf{e}^{\hat{1}} \wedge \mathbf{e}^0 + \frac{\lambda}{f_2(y)} \frac{\partial \ln \Phi}{\partial y} \mathbf{e}^2 \wedge \mathbf{e}^{\hat{2}} \wedge \mathbf{e}^0. \end{aligned} \quad (2.38)$$

This type of metrics includes the charged, rotating black strings discovered by [23] and Kaluza–Klein black holes in abelian heterotic supergravity, as we shall see in Sec. 4.

2.4.2. Type AII

In the case of $K_1 = K_2 = 0$, we can find another family of canonical frame $\{e_a\}$ such that

$$\begin{aligned} e_\mu &= \sqrt{\frac{X_\mu}{x_\mu^2 - x_\nu^2}} \frac{\partial}{\partial x_\mu}, \quad e_0 = \frac{\partial}{\partial \psi}, \\ e_{\hat{\mu}} &= \frac{1}{f_\mu \sqrt{(x_\mu^2 - x_\nu^2) X_\mu}} \left(x_\mu^2 \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} - N_\mu \frac{\partial}{\partial \psi} \right), \end{aligned} \quad (2.39)$$

where X_μ , N_μ and f_μ are unknown functions of one variable x_μ again. Then the corresponding metric is given by

$$\begin{aligned} g &= \frac{x^2 - y^2}{X(x)} dx^2 + \frac{y^2 - x^2}{Y(y)} dy^2 + (d\psi + \mathbf{W}_2)^2 \\ &+ \frac{f_1(x)^2 X(x)}{x^2 - y^2} (d\tau + y^2 d\sigma)^2 + \frac{f_2(y)^2 Y(y)}{y^2 - x^2} (d\tau + x^2 d\sigma)^2, \end{aligned} \quad (2.40)$$

where

$$\mathbf{W}_2 = \frac{N_1(x)}{x^2 - y^2} (d\tau + y^2 d\sigma) + \frac{N_2(y)}{y^2 - x^2} (d\tau + x^2 d\sigma). \quad (2.41)$$

The metric contains 6 unknown functions of one variable, $X(x)$, $Y(y)$, $N_1(x)$, $N_2(y)$, $f_1(x)$ and $f_2(y)$ and has three Killing vector fields $\partial/\partial\tau$, $\partial/\partial\sigma$ and $\partial/\partial\psi$. The torsion is given by

$$\begin{aligned} \mathbf{T} &= \left(1 - \frac{f_2(y)}{f_1(x)}\right) \frac{2x}{x^2 - y^2} \sqrt{\frac{Y(y)}{y^2 - x^2}} e^1 \wedge e^{\hat{1}} \wedge e^{\hat{2}} \\ &+ \left(1 - \frac{f_1(x)}{f_2(y)}\right) \frac{2y}{y^2 - x^2} \sqrt{\frac{X(x)}{x^2 - y^2}} e^2 \wedge e^{\hat{2}} \wedge e^{\hat{1}} \\ &+ \frac{1}{f_1(x)} \frac{\partial \Phi}{\partial x} e^1 \wedge e^{\hat{1}} \wedge e^0 + \frac{1}{f_2(y)} \frac{\partial \Phi}{\partial y} e^2 \wedge e^{\hat{2}} \wedge e^0. \end{aligned} \quad (2.42)$$

This type of metrics includes the charged, rotating Kaluza–Klein black holes in five-dimensional minimal supergravity theory discussed in the next section.

2.5. Type B metric

Let us consider the next case $M_1 = M_2 = 0$. This is an exceptional type appearing only when the torsion is present. In this case, we find a solution

$$e_\mu = \sqrt{\frac{X_\mu}{x_\mu^2 - x_\nu^2}} \left(\frac{\partial}{\partial x_\mu} - Z_\mu \frac{\partial}{\partial \psi} \right), \quad e_{\hat{\mu}} = \sqrt{\frac{Y_\mu}{x_\mu^2 - x_\nu^2}} \frac{\partial}{\partial y_\mu}, \quad e_0 = \frac{\partial}{\partial \psi}, \quad (2.43)$$

where X_μ , Y_μ and Z_μ are functions of two variables x_μ and y_μ . Rewriting $(x_1, x_2, y_1, y_2) = (x, y, \tau, \sigma)$, we obtain the corresponding metric by

$$\begin{aligned} g &= \frac{x^2 - y^2}{X_1(x, \tau)} dx^2 + \frac{y^2 - x^2}{X_2(y, \sigma)} dy^2 + \frac{x^2 - y^2}{Y_1(x, \tau)} d\tau^2 + \frac{y^2 - x^2}{Y_2(y, \sigma)} d\sigma^2 \\ &+ (d\psi + Z_1(x, \tau) dx + Z_2(y, \sigma) dy)^2. \end{aligned} \quad (2.44)$$

The metric contains 6 unknown functions of two variables: $X_1(x, \tau)$, $Y_1(x, \tau)$, $Z_1(x, \tau)$, $X_2(y, \sigma)$, $Y_2(y, \sigma)$ and $Z_2(y, \sigma)$. The torsion is given by

$$\begin{aligned} \mathbf{T} = & \frac{2x}{x^2 - y^2} \sqrt{\frac{X_2(y, \sigma)}{y^2 - x^2}} \mathbf{e}^1 \wedge \mathbf{e}^{\hat{1}} \wedge \mathbf{e}^{\hat{2}} + \frac{2y}{y^2 - x^2} \sqrt{\frac{X_1(x, \tau)}{x^2 - y^2}} \mathbf{e}^2 \wedge \mathbf{e}^{\hat{2}} \wedge \mathbf{e}^{\hat{1}} \\ & + \sqrt{\frac{X_1(x, \tau)Y_1(x, \tau)}{(x^2 - y^2)^2}} \frac{\partial Z_1(x, \tau)}{\partial \tau} \mathbf{e}^1 \wedge \mathbf{e}^{\hat{1}} \wedge \mathbf{e}^0 \\ & + \sqrt{\frac{X_2(y, \sigma)Y_2(y, \sigma)}{(y^2 - x^2)^2}} \frac{\partial Z_2(y, \sigma)}{\partial \sigma} \mathbf{e}^2 \wedge \mathbf{e}^{\hat{2}} \wedge \mathbf{e}^0 . \end{aligned} \quad (2.45)$$

2.6. Type C metric

In the last case, by virtue of symmetry between x_1 and x_2 , we can take $K_2 = 0$ and $M_2 = 0$ without loss of generality. The metric is given by

$$\begin{aligned} \mathbf{g} = & (x^2 - y^2) \left(\frac{dx^2}{X(x, \tau)} - \frac{dy^2}{Y(y)} + \Psi_1(x)^2 d\tau^2 + \Psi_2(y)^2 (-\Xi(x)d\tau + d\sigma)^2 \right) \\ & + (d\psi + \Omega_1(x, y)d\tau + \Omega_2(y)d\sigma)^2 , \end{aligned} \quad (2.46)$$

which contains five functions of single variable $\Xi(x)$, $Y(y)$, $\Psi_1(x)$, $\Psi_2(y)$, $\Omega_2(y)$ and two of two variables $X(x, \tau)$ and $\Omega_1(x, y)$.

3. Killing–Yano symmetry of Kaluza–Klein black holes in Einstein–Maxwell–Chern–Simons theory

In this section, we investigate Killing–Yano symmetry of Kaluza–Klein black holes in five-dimensional Einstein–Maxwell–Chern–Simons theory. The action of the theory consists of a (Lorentzian) metric $g_{\mu\nu}$ and a Maxwell field A_μ ,

$$S = \int * (R + \Lambda) - \frac{1}{2} * \mathbf{F} \wedge \mathbf{F} + \frac{\lambda_{cs}}{3\sqrt{3}} \mathbf{F} \wedge \mathbf{F} \wedge \mathbf{A} . \quad (3.1)$$

where $\mathbf{F} = d\mathbf{A}$ is a field strength of the Maxwell field, Λ is a cosmological constant and λ_{cs} is coupling constant of the Chern–Simons term. This is said to be the pure Einstein–Maxwell theory when $\lambda_{cs} = 0$ and the minimal supergravity when $\lambda_{cs} = 1$. The equations of motion are given by

$$R_{ab} + \frac{\Lambda}{3} g_{ab} = \frac{1}{2} \left(F_{ac} F_b{}^c - \frac{1}{6} g_{ab} F_{cd} F^{cd} \right) , \quad (3.2)$$

$$d * \mathbf{F} - \frac{\lambda_{cs}}{\sqrt{3}} \mathbf{F} \wedge \mathbf{F} = 0 . \quad (3.3)$$

3.1. Uplift of the Kerr–Newman solution

Many exact solutions of five-dimensional Einstein–Maxwell–Chern–Simons theory have already been discovered in literature. Of them, we focus especially on Kaluza–Klein type metrics,

$$\mathbf{g} = e^{-2\phi/\sqrt{3}} \mathbf{g}^{(4)} + e^{\phi/\sqrt{3}} (d\psi + \mathcal{W}_i dx^i)^2 , \quad (3.4)$$

with a Maxwell field written in the form

$$\mathbf{A} = \mathcal{A}_i dx^i + \rho d\psi , \quad (3.5)$$

where \mathcal{W}_i , \mathcal{A}_i , ϕ and ρ are functions of the coordinates x^i in four dimensions. Then we find that by setting

$$\phi = \rho = 0 , \quad \star \mathcal{G} = \frac{1}{\sqrt{3}} \mathcal{F} , \quad (3.6)$$

where \star represents the Hodge star with respect to $\mathbf{g}^{(4)}$, $\mathcal{G} = d\mathcal{W}$ and $\mathcal{F} = d\mathcal{A}$ are field strengths in four dimensions, the action (3.1) consistently reduces to the four-dimensional Einstein–Maxwell theory (e.g., see [26]). According to [27], this identification leads to an uplift of the Reissner–Nordström solution for an arbitrary value of the coupling λ_{cs} . This fact motivates us to consider S^1 bundle over the Kerr–Newman spacetime. That is, the ansatz is the following:

$$\begin{aligned} \mathbf{g} = & -\frac{\Delta}{\Sigma}(dt - a \sin^2 \theta d\phi)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} (adt - (r^2 + a^2)d\phi)^2 \\ & + \left(\frac{L}{2} d\psi - \frac{\alpha q r}{\Sigma} (dt - a \sin^2 \theta d\phi) - \frac{\beta q \cos \theta}{\Sigma} (adt - (r^2 + a^2)d\phi) \right)^2 , \end{aligned} \quad (3.7)$$

$$\mathbf{A} = -\frac{\gamma q r}{\Sigma} (dt - a \sin^2 \theta d\phi) - \frac{\delta q \cos \theta}{\Sigma} (adt - (r^2 + a^2)d\phi) , \quad (3.8)$$

and

$$\Delta = r^2 + a^2 + q^2 - 2mr , \quad \Sigma = r^2 + a^2 \cos^2 \theta . \quad (3.9)$$

If the Einstein–Maxwell–Chern–Simons theory is imposed, the equations of motion require the parameters α , β , γ and δ to satisfy some algebraic relations. The Einstein’s Eq. (3.2) yields

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 4 , \quad (3.10)$$

$$3(\alpha^2 - \beta^2) + \gamma^2 - \delta^2 = 0 , \quad (3.11)$$

$$3\alpha\beta + \gamma\delta = 0 , \quad (3.12)$$

which can be solved by

$$\alpha = \frac{\delta}{\sqrt{3}} , \quad \beta = -\frac{\gamma}{\sqrt{3}} , \quad \alpha^2 + \beta^2 = 1 . \quad (3.13)$$

These relations actually imply the conditions (3.6). Although the Maxwell’s Eq. (3.3) impose additional conditions

$$\alpha\gamma - \beta\delta - \frac{2\lambda_{cs}}{\sqrt{3}}\gamma\delta = 0 , \quad (3.14)$$

$$\alpha\delta + \beta\gamma + \frac{\lambda_{cs}}{\sqrt{3}}(\gamma^2 - \delta^2) = 0 , \quad (3.15)$$

the relations (3.13) automatically guarantee them when $\lambda_{cs} = 1$. Otherwise, we have the trivial solution $\alpha = \beta = \delta = \gamma = 0$. If we were to set $a = 0$ which corresponds to non-rotating (four-dimensional) spacetimes, Eq. (3.12) would have been absent and dynamical solutions could have been obtained for $\lambda_{cs} \neq 1$ [27]. In the presence of

rotation $a \neq 0$, only the minimal supergravity can accommodate the uplift within the present setup.

The metric describes charged, rotating Kaluza–Klein black holes when $\beta \neq 0$ and black strings when $\beta = 0$. The constant L represents the size of the extra dimension. The parameters m , a , q , α , γ and δ are related to five charges: mass M , angular momentum J^ψ and J^ϕ that are associated with the ψ and ϕ directions, electric charge Q , and magnetic flux Ψ . Thus obtained solution is trivial because this is just an uplift of the four-dimensional Kerr–Newman solution. However, as we will see below, this is an interesting example from the view point of generalised Killing–Yano symmetry.

3.2. Hidden symmetry

In this subsection, we show that the solutions obtained above fall into the general class of metrics derived in the previous section. Then, by writing down the Hamilton–Jacobi equation and Klein–Gordon equation on those spacetimes, we explicitly demonstrate the connection between the separability of these equations and the underlying Killing–Yano symmetry.

3.2.1. Killing–Yano symmetry

The metric (3.7) admits three Killing vectors $\partial/\partial\tau$, $\partial/\partial\sigma$ and $\partial/\partial\psi$. Besides them, we can find a rank-2 Killing–Stäckel tensor and a rank-2 generalised Killing–Yano tensor. To see this, it is helpful to use the coordinates

$$p = a \cos \theta, \quad \tau = t - a\phi, \quad \sigma = \frac{\phi}{a}, \quad (3.16)$$

which Carter introduced in [2] to study separability of the Hamilton–Jacobi equation for geodesics in the Kerr spacetime. The coordinate transformation (3.16) brings the metric (3.7) to a simple algebraical form

$$\begin{aligned} g = & \frac{r^2 + p^2}{Q} dr^2 + \frac{r^2 + p^2}{P} dp^2 \\ & - \frac{f_1^2 Q}{r^2 + p^2} (d\tau + p^2 d\sigma)^2 + \frac{f_2^2 P}{r^2 + p^2} (d\tau - r^2 d\sigma)^2 \\ & + \left(d\psi + \frac{N_1}{r^2 + p^2} (d\tau + p^2 d\sigma) + \frac{N_2}{r^2 + p^2} (d\tau - r^2 d\sigma) \right)^2, \end{aligned} \quad (3.17)$$

where

$$\begin{aligned} Q &= r^2 + a^2 + q^2 - 2mr, \quad P = a^2 - p^2, \\ N_1 &= -\alpha q r, \quad N_2 = -\beta q p, \quad f_1 = 1, \quad f_2 = 1. \end{aligned} \quad (3.18)$$

The inverse metric is given by

$$\begin{aligned} \left(\frac{\partial}{\partial s}\right)^2 = & -\frac{1}{f_1^2(r^2+p^2)\mathcal{Q}}\left(r^2\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma} - N_1\frac{\partial}{\partial\psi}\right)^2 \\ & + \frac{1}{f_2^2(r^2+p^2)\mathcal{P}}\left(p^2\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma} - N_2\frac{\partial}{\partial\psi}\right)^2 \\ & + \frac{\mathcal{Q}}{r^2+p^2}\left(\frac{\partial}{\partial r}\right)^2 + \frac{\mathcal{P}}{r^2+p^2}\left(\frac{\partial}{\partial p}\right)^2 + \left(\frac{\partial}{\partial\psi}\right)^2, \end{aligned} \quad (3.19)$$

Looking at (3.17), we notice that the metric is a Lorentzian counterpart of the type AII metric (2.40) obtained in Sec. 2.4. This means that (3.17) admits a rank-2 GK Y tensor. We may consider the metric with the functions of (3.18) replaced by arbitrary one variable functions $\mathcal{Q}(r)$, $\mathcal{P}(p)$, $N_1(r)$, $N_2(p)$, $f_1(r)$ and $f_2(p)$, which we call off-shell metric. By considering such an off-shell metric, we can deal with the metric (3.7) in a more algebraically general framework. For the off-shell metric (3.17), the canonical orthonormal frame is introduced as

$$\begin{aligned} e^1 &= \sqrt{\frac{r^2+p^2}{\mathcal{Q}}}dr, \quad e^{\hat{1}} = f_1\sqrt{\frac{\mathcal{Q}}{r^2+p^2}}(d\tau + p^2d\sigma), \\ e^2 &= \sqrt{\frac{r^2+p^2}{\mathcal{P}}}dp, \quad e^{\hat{2}} = f_2\sqrt{\frac{\mathcal{P}}{r^2+p^2}}(d\tau - r^2d\sigma), \\ e^0 &= d\psi + \frac{N_1}{r^2+p^2}(d\tau + p^2d\sigma) + \frac{N_2}{r^2+p^2}(d\tau - r^2d\sigma). \end{aligned} \quad (3.20)$$

With respect to this canonical frame, we can easily write down the metric and the rank-2 GK Y tensor as

$$\mathbf{g} = e^1e^1 - e^{\hat{1}}e^{\hat{1}} + e^2e^2 + e^{\hat{2}}e^{\hat{2}} + e^0e^0, \quad (3.21)$$

$$\mathbf{f} = p e^1 \wedge e^{\hat{1}} + r e^2 \wedge e^{\hat{2}}. \quad (3.22)$$

From (2.9), we obtain a rank-2 Killing–Stäckel tensor

$$\mathbf{K} = p^2(e^1e^1 - e^{\hat{1}}e^{\hat{1}}) - r^2(e^2e^2 + e^{\hat{2}}e^{\hat{2}}), \quad (3.23)$$

which is given in terms of the coordinate basis by

$$\begin{aligned} K^{ab}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b} = & -\frac{p^2}{f_1^2(r^2+p^2)\mathcal{Q}}\left(r^2\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma} - N_1\frac{\partial}{\partial\psi}\right)^2 \\ & + \frac{r^2}{f_2^2(r^2+p^2)\mathcal{P}}\left(p^2\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma} - N_2\frac{\partial}{\partial\psi}\right)^2 \\ & + \frac{p^2\mathcal{Q}}{r^2+p^2}\left(\frac{\partial}{\partial r}\right)^2 - \frac{r^2\mathcal{P}}{r^2+p^2}\left(\frac{\partial}{\partial p}\right)^2. \end{aligned} \quad (3.24)$$

3.2.2. Separation of variables in the Hamilton–Jacobi equation

The existence of rank-2 Killing–Stäckel tensors is in general related to separation of variables in Hamilton–Jacobi equations for geodesics

$$g^{ab}\partial_a S \partial_b S = -m^2. \quad (3.25)$$

For the off-shell metric (3.17), since the inverse metric is given by (3.19), the Hamilton–Jacobi equation for geodesics (3.25) can be solved by separation of variables with a function

$$S = R(r) + \Theta(p) + \pi_\tau \tau + \pi_\sigma \sigma + \pi_\psi \psi , \quad (3.26)$$

where π_τ , π_σ and π_ψ are arbitrary constants, and the functions $R(r)$ and $\Theta(p)$ satisfy the ordinary differential equations

$$\left(\frac{dR}{dr}\right)^2 - \left(\frac{W_r}{f_1 Q}\right)^2 - \frac{V_r}{Q} = 0 , \quad \left(\frac{d\Theta}{dp}\right)^2 + \left(\frac{W_p}{f_2 \mathcal{P}}\right)^2 - \frac{V_p}{\mathcal{P}} = 0 \quad (3.27)$$

with the potentials including a separation constant κ ,

$$\begin{aligned} W_r &= r^2 \pi_\tau + \pi_\sigma - N_1 \pi_\psi , & V_r &= -(\pi_\psi^2 + m^2) r^2 + \kappa , \\ W_p &= p^2 \pi_\tau - \pi_\sigma - N_2 \pi_\psi , & V_p &= -(\pi_\psi^2 + m^2) p^2 - \kappa . \end{aligned} \quad (3.28)$$

The constant κ is given, by eliminating $-m^2$, as

$$\kappa = \frac{p^2}{r^2 + p^2} \left(Q \Pi_r^2 - \frac{W_r^2}{f_1^2 Q} \right) - \frac{r^2}{r^2 + p^2} \left(\mathcal{P} \Pi_p^2 - \frac{W_p^2}{f_2^2 \mathcal{P}} \right) . \quad (3.29)$$

where $\Pi_a = \partial_a S$ is the canonical momentum given by

$$\Pi_r = \pm \sqrt{\left(\frac{W_r}{f_1 Q}\right)^2 + \frac{V_r}{Q}} , \quad \Pi_p = \pm \sqrt{-\left(\frac{W_p}{f_2 \mathcal{P}}\right)^2 + \frac{V_p}{\mathcal{P}}} \quad (3.30)$$

and $\Pi_\tau = \pi_\tau$, $\Pi_\sigma = \pi_\sigma$ and $\Pi_\psi = \pi_\psi$.

We have now four constants of motion. Three of them Π_τ , Π_σ and Π_ψ are associated with the Killing vectors $\partial/\partial\tau$, $\partial/\partial\sigma$ and $\partial/\partial\psi$. The Killing–Stäckel tensor K_{ab} is responsible for the other one κ . In fact, from the equation (3.24), one can easily confirm that

$$\kappa = K^{ab} \Pi_a \Pi_b , \quad (3.31)$$

which is precisely the general relation (2.8). Its constancy is in general guaranteed by the commutativity with the Hamiltonian $H = g^{ab} \Pi_a \Pi_b$ under the Poisson bracket, namely $\{H, \kappa\} = 0$. Thus we have found that the charged, rotating Kaluza–Klein black hole spacetime (3.7) does possess the separable structure for the Hamilton–Jacobi equation and its separation constant is indeed related to the underlying Killing–Yano symmetry that generates the rank-2 Killing–Stäckel tensor necessary for separation of variables.

3.2.3 Separation of variables in the Klein–Gordon equation

The massive Klein–Gordon equation is given by

$$(\square + m^2) \Psi = 0 . \quad (3.32)$$

With the help of $\square \Psi = (1/\sqrt{-g}) \nabla_a (\sqrt{-g} g^{ab} \nabla_b \Psi)$, we have

$$\begin{aligned} & \left[\frac{1}{f_1} \frac{\partial}{\partial r} \left(f_1 Q \frac{\partial}{\partial r} \right) + \frac{1}{f_2} \frac{\partial}{\partial p} \left(f_2 \mathcal{P} \frac{\partial}{\partial p} \right) - \frac{1}{f_1^2 Q} \left(r^2 \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \sigma} - N_1 \frac{\partial}{\partial \psi} \right)^2 \right. \\ & \left. + \frac{1}{f_2^2 \mathcal{P}} \left(p^2 \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} - N_2 \frac{\partial}{\partial \psi} \right)^2 + (r^2 + p^2) \left(\frac{\partial^2}{\partial \psi^2} + m^2 \right) \right] \Psi = 0 . \end{aligned} \quad (3.33)$$

This equation is separable with a function

$$\Psi = R(r)\Theta(p)e^{\pi_\tau\tau+\pi_\sigma\sigma+\pi_\psi\psi} , \quad (3.34)$$

where π_τ , π_σ and π_ψ are constants again, and the functions $R(r)$ and $\Theta(p)$ satisfy the ordinary differential equations

$$\frac{1}{f_1}\frac{d}{dr}\left(f_1\mathcal{Q}\frac{dR}{dr}\right) - \frac{W_r^2}{f_1^2\mathcal{Q}} - V_r = 0 , \quad \frac{1}{f_2}\frac{d}{dp}\left(f_2\mathcal{P}\frac{d\Theta}{dp}\right) + \frac{W_p^2}{f_2^2\mathcal{P}} - V_p = 0 \quad (3.35)$$

with the potentials given by (3.28) including the separation constant κ . By eliminating m^2 , the constant κ is this time given by

$$\begin{aligned} \kappa = & \frac{p^2}{r^2 + p^2} \left[\frac{1}{f_1}\frac{d}{dr}\left(f_1\mathcal{Q}\frac{dR}{dr}\right) - \frac{W_r^2}{f_1^2\mathcal{Q}} \right] \\ & + \frac{r^2}{r^2 + p^2} \left[\frac{1}{f_2}\frac{d}{dp}\left(f_2\mathcal{P}\frac{d\Theta}{dp}\right) + \frac{W_p^2}{f_2^2\mathcal{P}} \right] . \end{aligned} \quad (3.36)$$

Again, it is straightforward to check that this constant coincides with the one following from the general property of the symmetry operator (2.10) together with the Killing–Stäckel tensor (3.23), that is, we have

$$\hat{K}\Psi \equiv \nabla_a K^{ab} \nabla_b \Psi = \kappa \Psi . \quad (3.37)$$

This is the relationship between Killing–Stäckel tensor and separability of the Klein–Gordon equation.

3.3. More general solution

Our purpose here is to construct the most general solution of the Einstein–Maxwell–Chern–Simons theory, namely the equations (3.2) and (3.3), for the type AII off-shell metric (3.17). We adopt the ansatz that identifies torsion with the flux by

$$\mathbf{T} = \frac{1}{\sqrt{3}} * \mathbf{F} . \quad (3.38)$$

Under the assumption (3.38), the Maxwell–Chern–Simons equation (3.3) and the Bianchi identity, $d\mathbf{F} = 0$, are written as

$$d\mathbf{T} - \lambda_{cs}(*\mathbf{T}) \wedge (*\mathbf{T}) = 0 , \quad d*\mathbf{T} = 0 . \quad (3.39)$$

Substituting the expression for the torsion (2.42), the first equation requires both $f_1 = f_2$ and $\lambda_{cs} = 1$. Since f_1 and f_2 are one variable functions of only r and p respectively, we find that $f_1 = f_2$ must be constant and then it can be absorbed in N_μ via rescaling of τ and σ . Thus, we may set $f_1 = f_2 = 1$. The restriction to $\lambda_{cs} = 1$ corresponds to the minimal supergravity. The second equation then solves as

$$N_1 = \tilde{a}r^2 + b_1r , \quad N_2 = \tilde{a}p^2 + b_2p , \quad (3.40)$$

where \tilde{a} , b_1 and b_2 are constants. Note that \tilde{a} is a gauge parameter which can be eliminated by gauge transformation of ψ . We set it to be zero. These conditions render

many components of the Einstein equations trivial. We need $(0, 0)$ component to derive $\Lambda = 0$. Then $(3, 3)$ and $(4, 4)$ components determine \mathcal{Q} and \mathcal{P} as

$$\mathcal{Q} = \tilde{c}r^2 + m_1r + q_1, \quad \mathcal{P} = -\tilde{c}p^2 + m_2p + q_2, \quad (3.41)$$

where \tilde{c}, m_1, m_2, q_1 and q_2 are constants which satisfy

$$q_1 - q_2 = b_1^2 + b_2^2. \quad (3.42)$$

Again the gauge freedom enables us to rescale \tilde{c} to be 1 so that we have obtained a five-parameter family of solutions.

In order to compare it with the Kaluza–Klein black hole solution in Sec. 3.1, we make the coordinate transformation (3.16) and set $\mu = m_2/a$ and $q_0 = q_2/a^2$. The obtained solution is written as

$$\begin{aligned} \mathbf{g} = & -\frac{\Delta}{\Sigma} (dt - a \sin^2 \theta d\phi)^2 + \frac{\Sigma}{\Delta} dr^2 + \frac{\Sigma \sin^2 \theta}{\sin^2 \theta + \mu \cos \theta + q_0 - 1} d\theta^2 \\ & + \frac{\sin^2 \theta + \mu \cos \theta + q_0 - 1}{\Sigma} (adt - (r^2 + a^2) d\phi)^2 \\ & + \left(d\psi + \frac{b_1 r}{\Sigma} (dt - a \sin^2 \theta d\phi) + \frac{b_2 \cos \theta}{\Sigma} (adt - (r^2 + a^2) d\phi) \right)^2, \\ \mathbf{A} = & -\frac{\sqrt{3}b_1 r}{\Sigma} (dt - a \sin^2 \theta d\phi) + \frac{\sqrt{3}b_2 \cos \theta}{\Sigma} (adt - (r^2 + a^2) d\phi), \end{aligned} \quad (3.43)$$

where

$$\Delta = r^2 + m_1 r + a^2 q_0 + b_1^2 + b_2^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta. \quad (3.44)$$

It can be seen that the metric is the uplift of the Kerr–Newman–NUT solution [2]. The result of Sec. 3.1 is included as a special case $\mu = 0$ and $q_0 = 1$.

4. Construction of Kaluza–Klein black holes in heterotic supergravity

In this section we consider abelian heterotic supergravity in five dimensions, which is the low-energy effective theory of heterotic string theory. The string-frame action consists of a (Lorentzian) metric $g_{\mu\nu}$, scalar field φ , $U(1)$ gauge potential A_μ and 2-form potential $B_{\mu\nu}$,

$$S = \int e^\varphi \left(R + *d\varphi \wedge d\varphi - *\mathbf{F} \wedge \mathbf{F} - \frac{1}{2} * \mathbf{H} \wedge \mathbf{H} \right), \quad (4.1)$$

where $\mathbf{F} = d\mathbf{A}$ and $\mathbf{H} = d\mathbf{B} - \mathbf{A} \wedge d\mathbf{A}$. The equations of motion are

$$R_{ab} - \nabla_a \nabla_b \varphi - F_a{}^c F_{bc} - \frac{1}{4} H_a{}^{cd} H_{bcd} = 0, \quad (4.2)$$

$$d(e^\varphi * \mathbf{F}) - e^\varphi * \mathbf{H} \wedge \mathbf{F} = 0, \quad (4.3)$$

$$d(e^\varphi * \mathbf{H}) = 0, \quad (4.4)$$

$$R - (\nabla\varphi)^2 - 2\nabla^2\varphi - \frac{1}{2}F^2 - \frac{1}{12}H^2 = 0. \quad (4.5)$$

We start from reviewing a known black string solution of this theory, revealing its Killing–Yano symmetry. Then using the general form of metric obtained in section 2, we construct a class of charged, rotating Kaluza–Klein black holes.

4.1. Hidden symmetry of the Mahapatra's solution

Using the technique of S. F. Hassan and A. Sen [28], a charged, rotating black string solution with four parameters $(m, a, \delta_1, \delta_2)$ was constructed by S. Mahapatra [23]. The solution is given by

$$\mathbf{g} = -\frac{\Delta}{\Sigma} \left(dt - a \sin^2 \theta d\phi - \hat{W} \right)^2 + \frac{a^2 \sin^2 \theta}{\Sigma} \left(dt - \frac{r^2 + a^2}{a} d\phi - \mathbf{W} \right)^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(d\psi - \frac{\beta}{1-\alpha} \mathbf{W} \right)^2, \quad (4.6)$$

$$\mathbf{A} = -\frac{\gamma}{1-\alpha} \mathbf{W}, \quad (4.7)$$

$$\mathbf{B} = \left(-dt + \frac{\beta}{1-\alpha} d\psi \right) \wedge \mathbf{W}, \quad (4.8)$$

$$e^\varphi = 1 + \frac{-mr(1-\alpha)}{\Sigma}, \quad (4.9)$$

where

$$\Delta = r^2 + a^2 - 2mr, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad (4.10)$$

$$\mathbf{W} = \frac{-mr(1-\alpha)}{\Sigma - mr(1-\alpha)} (dt - a \sin^2 \theta d\phi). \quad (4.11)$$

The parameters α , β and γ are required to satisfy the relation $\alpha^2 = 1 + \beta^2 + \gamma^2$, which can be written by two parameters δ_1 and δ_2 as

$$\alpha = \cosh \delta_1 \cosh \delta_2, \quad \beta = \cosh \delta_1 \sinh \delta_2, \quad \gamma = \sinh \delta_1. \quad (4.12)$$

The field strengths \mathbf{F} and \mathbf{H} are easily computed as

$$\mathbf{F} = -\frac{\gamma}{1-\alpha} d\mathbf{W}, \quad \mathbf{H} = \boldsymbol{\eta} \wedge d\mathbf{W}, \quad (4.13)$$

where

$$\begin{aligned} \boldsymbol{\eta} &= dt - \frac{\beta}{1-\alpha} d\psi - \frac{\gamma^2}{(1-\alpha)^2} \mathbf{W} \\ &= \frac{\Delta}{\Sigma} (dt - a \sin^2 \theta d\phi - \mathbf{W}) - \frac{a^2 \sin^2 \theta}{\Sigma} \left(dt - \frac{r^2 + a^2}{a} d\phi - \mathbf{W} \right) \\ &\quad - \frac{\beta}{1-\alpha} \left(d\psi - \frac{\beta}{1-\alpha} \mathbf{W} \right). \end{aligned} \quad (4.14)$$

4.1.1. Killing-Yano symmetry

Let us unveil the Killing-Yano symmetry of the Mahapatra's spacetime. Performing the coordinate transformation (3.16) again, the metric (4.6) can be written in a simple algebraical form

$$\mathbf{g} = -\frac{f_1^2 \mathcal{Q}}{r^2 + p^2} \left(d\tau + p^2 d\sigma - \mathbf{W} \right)^2 + \frac{f_2^2 \mathcal{P}}{r^2 + p^2} \left(d\tau - r^2 d\sigma - \mathbf{W} \right)^2 + \frac{r^2 + p^2}{\mathcal{Q}} dr^2 + \frac{r^2 + p^2}{\mathcal{P}} dp^2 + \left(d\psi + \lambda \mathbf{W} \right)^2, \quad (4.15)$$

where

$$\begin{aligned}
\mathbf{W} &= \frac{1}{\Phi} \left(\frac{N_1}{\Sigma} (d\tau + p^2 d\sigma) + \frac{N_2}{\Sigma} (d\tau - r^2 d\sigma) \right), \\
\Phi &= 1 + \frac{N_1}{\Sigma} + \frac{N_2}{\Sigma}, \quad \Sigma = r^2 + p^2, \\
\mathcal{Q} &= r^2 - 2mr + a^2, \quad \mathcal{P} = a^2 - p^2, \quad N_1 = -mr(1 - \alpha), \\
N_2 &= 0, \quad f_1 = 1, \quad f_2 = 1, \quad \lambda = -\frac{\beta}{1 - \alpha}.
\end{aligned} \tag{4.16}$$

We observe that this metric falls into the family AI (2.36) in the general classification of Sec. 2. Similarly to the previous section, we consider hidden symmetry for the off-shell metric (4.15) with \mathcal{Q} , \mathcal{P} , N_1 , N_2 , f_1 , f_2 and λ replaced by unknown functions $\mathcal{Q}(r)$, $\mathcal{P}(p)$, $N_1(r)$, $N_2(p)$, $f_1(r)$ and $f_2(p)$ and an arbitrary constant λ . For the off-shell metric (4.15), the canonical frame is introduced as

$$\begin{aligned}
e^1 &= \sqrt{\frac{r^2 + p^2}{\mathcal{Q}}} dr, \quad e^{\hat{1}} = f_1 \sqrt{\frac{\mathcal{Q}}{r^2 + p^2}} (d\tau + p^2 d\sigma - \mathbf{W}), \\
e^2 &= \sqrt{\frac{r^2 + p^2}{\mathcal{P}}} dp, \quad e^{\hat{2}} = f_2 \sqrt{\frac{\mathcal{P}}{r^2 + p^2}} (d\tau - r^2 d\sigma - \mathbf{W}), \\
e^0 &= d\psi + \lambda \mathbf{W}.
\end{aligned} \tag{4.17}$$

With respect to this orthonormal frame, the metric and the rank-2 generalised Killing–Yano tensor are written as (3.21) and (3.22). Furthermore, we also obtain a Killing–Stäckel tensor of the form (3.23). Since the inverse metric is given by

$$\begin{aligned}
\left(\frac{\partial}{\partial s} \right)^2 &= -\frac{1}{f_1^2(r^2 + p^2)\mathcal{Q}} \left((r^2 + N_1) \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \sigma} - \lambda N_1 \frac{\partial}{\partial \psi} \right)^2 \\
&\quad + \frac{1}{f_2^2(r^2 + p^2)\mathcal{P}} \left((p^2 + N_2) \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} - \lambda N_2 \frac{\partial}{\partial \psi} \right)^2 \\
&\quad + \frac{\mathcal{Q}}{r^2 + p^2} \left(\frac{\partial}{\partial r} \right)^2 + \frac{\mathcal{P}}{r^2 + p^2} \left(\frac{\partial}{\partial p} \right)^2 + \left(\frac{\partial}{\partial \psi} \right)^2,
\end{aligned} \tag{4.18}$$

the contravariant Killing–Stäckel tensor can be written as

$$\begin{aligned}
K^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} &= -\frac{p^2}{f_1^2(r^2 + p^2)\mathcal{Q}} \left((r^2 + N_1) \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \sigma} - \lambda N_1 \frac{\partial}{\partial \psi} \right)^2 \\
&\quad + \frac{r^2}{f_2^2(r^2 + p^2)\mathcal{P}} \left((p^2 + N_2) \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma} - \lambda N_2 \frac{\partial}{\partial \psi} \right)^2 \\
&\quad + \frac{p^2 \mathcal{Q}}{r^2 + p^2} \left(\frac{\partial}{\partial r} \right)^2 + \frac{r^2 \mathcal{P}}{r^2 + p^2} \left(\frac{\partial}{\partial p} \right)^2.
\end{aligned} \tag{4.19}$$

4.1.2. Separation of variables in the Hamilton–Jacobi equation

For the off-shell metric (4.15), the geodesic equation (3.25) can be solved by separation of variables with a function (3.26) and obtain the ordinary differential equations (3.27) with the potentials

$$W_r = (r^2 + N_1)\pi_\tau + \pi_\sigma - \lambda N_1 \pi_\psi, \quad V_r = -(\pi_\psi^2 + m^2)r^2 + \kappa,$$

$$W_p = (p^2 + N_2)\pi_\tau - \pi_\sigma - \lambda N_2 \pi_\psi , \quad V_p = -(\pi_\psi^2 + m^2)p^2 - \kappa , \quad (4.20)$$

where the separation constant κ is given by (3.29) with the above potentials. Indeed, since this constant is related to the Killing–Stäckel tensor (4.19) with the general formula (3.31), we find that separation of variables for the geodesic equation in the Mahapatra’s spacetime is underwritten by the existence of the generalised Killing–Yano symmetry.

4.1.3. Separation of variables in the Klein–Gordon equation

In contrast to the case of minimal supergravity black holes, the Klein–Gordon equation (3.32) for the off-shell metric (4.15) does not separate. On the other hand, the deformed Klein–Gordon equation

$$(\square + m^2)\Psi - (\nabla_a \varphi)(\nabla^a \Psi) = 0 \quad (4.21)$$

does with $\varphi = \ln \Phi$. Eq. (4.21) is equivalent to Klein–Gordon equation in Einstein frame,

$$(\square_E + m^2)\Psi = 0 , \quad (4.22)$$

where \square_E is the d’Alembertian with respect to the Einstein-frame metric $\mathbf{g}_E = \varphi^{2/3}\mathbf{g}$. Namely, separation of variables for the Klein–Gordon equation naturally occurs in the Einstein frame. The similar situation was seen in the Kerr–Sen black holes [21].

4.2. Kaluza–Klein black hole solutions

We attempt to find the general solutions of the equations of motion (4.2)–(4.5) which take the form of type AI metric (4.15), under the ansatz for the matter fields

$$\varphi = \ln \Phi , \quad \mathbf{F} = c_F d\mathbf{W} , \quad \mathbf{H} = c_H \mathbf{T} , \quad (4.23)$$

where \mathbf{T} is the type AI torsion (2.38). For a non-trivial solution, equation (4.3) requires

$$c_H = 1 . \quad (4.24)$$

Under this condition, the rest of (4.3) become dependent on the dynamical equations of \mathbf{H} (4.4). Combined with the diagonal part of the Einstein equations (4.2), one can derive

$$f_1 = f_2 , \quad \text{or} \quad f_1 = \tilde{f}x^2, \quad f_2 = \tilde{f}y^2 , \quad (4.25)$$

where \tilde{f} is constant. Since the latter is not consistent with the off-diagonal terms of the Einstein equations, we focus on the former and set $f_1 = f_2 = 1$ by redefining the coordinates as before. The remaining component of (4.4) and the (3, 4)-component of (4.2) lead to

$$N_1 = \tilde{a}r^2 + b_1r, \quad N_2 = \tilde{a}p^2 + b_2p. \quad (4.26)$$

Finally, the consistency condition $d\mathbf{H} = -\mathbf{F} \wedge \mathbf{F}$ determines \mathcal{P} and \mathcal{Q} as

$$\mathcal{P}(p) - \mathcal{Q}(r) = -(c_F^2 + \lambda^2)(r^2 + p^2) + c_\Phi(r^2 + p^2)\Phi. \quad (4.27)$$

(4.25), (4.26) and (4.27) are sufficient to guarantee that all the remaining equations are satisfied. By means of the physically irrelevant rescaling of Φ , one can choose $\tilde{a} = 0$ (note our definition of Φ (4.16) contains the normalised constant term) and derive

$$\begin{aligned}\mathcal{Q} &= (c_F^2 + \lambda^2 - c_\Phi) r^2 + b_1 c_\Phi r + c_0, \\ \mathcal{P} &= -(c_F^2 + \lambda^2 - c_\Phi) p^2 + b_2 c_\Phi p + c_0.\end{aligned}\tag{4.28}$$

The overall factor of \mathcal{P} and \mathcal{Q} can be gauged away, leaving a family of solutions with five parameters $(c_0, c_F, c_\Phi, b_1, b_2)$, describing charged, rotating black holes and black strings.

5. Summary and Discussions

Firstly, we have classified five-dimensional metrics admitting a rank-2 generalised Killing–Yano tensor under the assumption that its eigenvector associated with the zero eigenvalue is a Killing vector field. The metrics have been classified into three types A, B and C in general, and local expressions of the corresponding metrics have been obtained explicitly. One of the open problems is to classify them without assuming the Killing vector, in the presence of torsion. In this case, the large number of unknown variables arising from the components of $\boldsymbol{\xi}$ makes it difficult to solve the integrability conditions. However, even if it is not possible to obtain a general classification, it would be of a great interest as an attempt to find spacetimes of cohomogeneity three.

We also have demonstrated separability structures of the Hamilton–Jacobi and Klein–Gordon equations for some known charged, rotating Kaluza–Klein black hole and black string solutions in the five-dimensional minimal supergravity as well as heterotic supergravity. We have found that those spacetimes fall into the class of the type A metric and the separability structure is indeed related to the underlying generalised Killing–Yano symmetry. Hence, it would be also interesting to study separability of the Dirac equation and other test field equations such as Maxwell’s equations in those spacetimes. In our calculation, the torsion tensors associated with the generalised Killing–Yano symmetry have been identified with the matter fields as $\boldsymbol{T} = *\boldsymbol{F}/\sqrt{3}$ in the five-dimensional minimal supergravity and as $\boldsymbol{T} = \boldsymbol{H}$ in the abelian heterotic supergravity, which is analogous to the asymptotically flat black hole spacetimes. In the limit of vanishing torsion, both of them reduce to the Kerr string solution.

The obtained classification provides an alternative to various approaches for finding new exact solutions. In fact, using the type A metrics derived in Sec. 2, we have constructed the general solutions describing charged, rotating Kaluza–Klein black holes in the five-dimensional minimal supergravity and abelian heterotic supergravity. Although we have concentrated on the type A metrics in this paper since our primary aim has been the application to Kaluza–Klein black holes, type B and C metrics also offer possibilities to find novel exact solutions, which can be interesting since they exist only when the torsion is present. It should also be commented that all the calculations in this paper can be generalised to odd dimensions higher than five. It would enable us to seek uplift of charged, rotating black hole solutions to higher dimensions.

In [29], the existence of the ordinary Killing–Yano tensors was investigated on nearly Kähler manifolds and on manifolds with a weak G_2 -structure. The Killing–Yano equations on manifolds with G-structure were investigated in the absence [30] and presence [31] of torsion. The generalised Killing–Yano tensors are also related to Kähler manifolds in even dimensions and Sasaki manifolds in odd dimensions. It was demonstrated in [22] that Kähler manifolds studied by [33] admit rank-2 generalised conformal Killing–Yano tensors. In odd dimensions, a concrete example of the Killing–Yano tensor was constructed [32] on Sasaki manifolds studied by [34]. Moreover, a notion of deformed Sasaki manifolds in the presence of torsion was introduced by [25] and the authors have shown an example admitting a rank-3 generalised closed conformal Killing–Yano tensor. It can be shown that the Sasaki manifolds with torsion discussed in [25] also take the form of the type A metrics in our classification. The present work might be useful to obtain further examples of Sasaki manifolds with torsion.

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